Minimum Uncertainty Frequency Estimation using Past Measurements for Various Power-Law Noise Types

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***Abstract*—We present a method for current frequency estimation with minimum uncertainty using the past measurements. Estimated frequency is calculated as a weighted sum of the frequencies of the previous measurements and the weights are optimized to minimize the uncertainty. Uncertainties and optimum weights are calculated for various power-law noise types. For a case of a next step frequency estimation with N previous consecutive measurements, it is shown that only three previous measurements are enough to reduce the uncertainty to less than 5 % over the achievable minimum uncertainty for flicker FM. For white FM noise, we need to use 10 previous measurements to reach the same result. For a random-walk FM noise type, the uncertainty becomes less than 1 % over the limit value with only two measurements. Simple method to predict the frequency and phase for a drifting timescale is proposed.**

I. INTRODUCTION

Frequency prediction of a timescale from a clock or from an ensemble of clocks is necessary in many cases. Many of the ensemble timescale algorithms have procedures in them to estimate the frequency and phase of contributing clocks for the next step generation of timescale. Sometimes the ensemble timescale itself needs to be steered to some reference timescale to insure some pre-set time and/or frequency difference between the two. To do this, the estimation of the frequency and phase difference for the next step in free run status is needed to steer correctly to minimize the difference.

In many cases, the frequency is estimated adopting exponential filter to the previous frequency data. However, this does not always result in minimum uncertainty of estimation, which will be shown in this paper. In order to estimate an optimum frequency, we should first know theoretically the uncertainty itself when two or more measurements are used in the estimation. Then we can find optimum ways of treating the measured frequencies to get minimum uncertainty. A paper on this theory was published by Yu et al. [1] extending the work by Douglas and Boulanger [2]. We have developed a simple method to calculate the uncertainty of a frequency comparison in the presence of arbitrary configurations of dead time and measurement interval offset. More detailed description on the history of related theoretical studies can be found in [1].

In this paper, we will consider only the estimation of a frequency and phase for the next step with the previous consecutive measurements without any dead time. First, we will consider the timescale with no drift. The uncertainty and optimum weights will be obtained and their characteristics will be analyzed. The method to deal with the timescale with known drift rate will also be explained.

II. THEORY

Since, the details can be found in [1], brief introduction of the theoretical approach will be given in this paper. Assume the live frequency measurements in an interval A are performed at N different times. We can estimate the frequency at another time interval B by calculating a weighted mean of the frequencies using all the live measurements. The uncertainty U can be expressed as the mean-square difference between these two frequencies.

$$U^2 = \left\langle (\sum_{i=1}^{N} a_i y_i - y_T)^2 \right\rangle, \tag{1}$$

where y_i and a_i are the i-th measured frequency and normalized weight, respectively, in the interval A, and y_T is the unknown 'true' frequency for an interval B. It can easily be shown that (1) can be expanded into (2). We assume that the signal is stationary and interchange of the order of the integration in calculating autocovariance gives the same value. This step allows the results of [2] to be extended to the case with distributed live measurements:

$$U^2 = \sum_{i=1}^{N} a_i \left\langle (y_i - y_T)^2 \right\rangle - \frac{1}{2} \sum_{i,j=1}^{N} a_i a_j \left\langle (y_i - y_j)^2 \right\rangle. \tag{2}$$

As seen in (2), the uncertainty variance is the weighted sum of variances of each live interval to transfer to interval B, minus the weighted sum of covariance terms of all possible combinations between two live measurements. The uncertainty can easily be calculated with proper assignment of the weights a_i and the analytical solutions for the uncertainties $\langle (y_i - y_T)^2 \rangle$ and $\langle (y_i - y_j)^2 \rangle$ found in [2]. We can simply use the same weights for all the live measurements, but this will not produce the optimum frequency estimate with the minimum uncertainty. Given any configuration of live measurements, weights to minimize the uncertainty can be calculated by the Lagrangian multiplier method. We define the function F by

$$F = U^2 + \lambda(\sum_{i=1}^{N} a_i - 1) . \tag{3}$$

By use of (2), the minimum of U^2 is reached when

$$\frac{\partial F}{\partial a_i} = \langle (y_i - y_T)^2 \rangle - \sum_{j=1}^{N} a_j \langle (y_i - y_j)^2 \rangle + \lambda = 0 \tag{4-1}$$

$$\frac{\partial F}{\partial \lambda} = \sum_{i=1}^{N} a_i - 1 = 0. \tag{4-2}$$

Optimum weights for minimizing the uncertainty can easily be calculated by solving the above (N+1) linear equations with (N+1) variables. As is well known for the specific case of white frequency modulation (FM) noise, the solution shows that only the total measurement live time matters, and the weight of each measurement is proportional to the measurement interval, i.e., inversely proportional to the Allan variance, as follows.

$$a_i = \tau_i / \sum_{j=1}^{N} \tau_j = (1/\sigma_y^2(\tau_i)) / \sum_{j=1}^{N} (1/\sigma_y^2(\tau_j)) , \tag{5}$$

where τ_i is the i-th measurement interval. However, the optimum weights do not exactly follow (5) when there are other power law noise types.

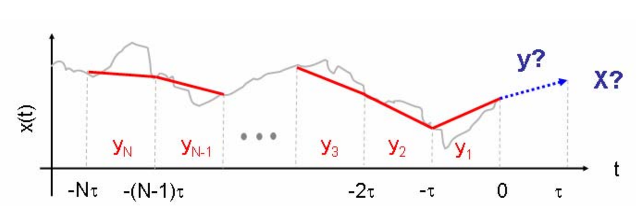

Figure 1. Schematic diagram of estimation of a frequency and phase with the previous N consecutive measurements.

III. PREDICTIONN OF FREQUENCY AND PHASE WITH MINIMUM UNCERTAINTY

Since we are interested in the estimation of a frequency for the next step with the previous consecutive measurements without any dead time, the phase estimation is done as a result.

As shown in Fig. 1, we want to estimate the frequency and phase for the next step with the real measurements done until 'now'. We assign the most recent measurement the index 1 and the oldest, N. All the measurements and prediction are assumed to be done for the same interval. We will consider first a timescale without drift and then show how to deal with a drifting timescale.

A. For a timescale without drift

Theory explained above can directly be used in this case. Fig. 2 shows the ratio of the uncertainty to the Allan deviation at a sampling time of measurement interval, for various numbers of measurements used in prediction, for different noise types. We can say that it is harder to predict the frequency for a random-walk noise case compared to that for a white FM one. The uncertainty shows a limiting value below which we cannot reduce any smaller. These rations are 1.00 for white FM, 1.30 for flicker FM, and 1. 37 for random-walk FM noise, respectively. Another interesting result is that only a small number of measurements are enough to reach close to the minimum achievable uncertainty. Only three previous measurements are enough to reduce the uncertainty to less than 5 % over the achievable minimum uncertainty for flicker FM. For white FM noise, we need to use 10 previous measurements to reach the same result. Even remarkable case is shown for a random-walk FM noise type, the uncertainty becomes less than 1 % over the limit value with only two measurements.

Fig. 3 shows the weights when we use five previous measurements for flicker FM and random-walk FM noise types. Weights for all the measurements are the same for white FM case, regardless of when the measurements are dome. As can be expected for non-white noise types, weight for the oldest measurement marked 2nd largest value for flicker

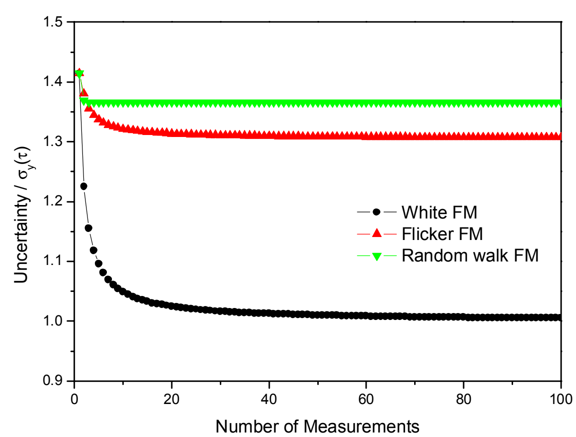

Figure 2. Uncertainty to Allan devialtion ratio vs. number of measurements used for three different noise types.

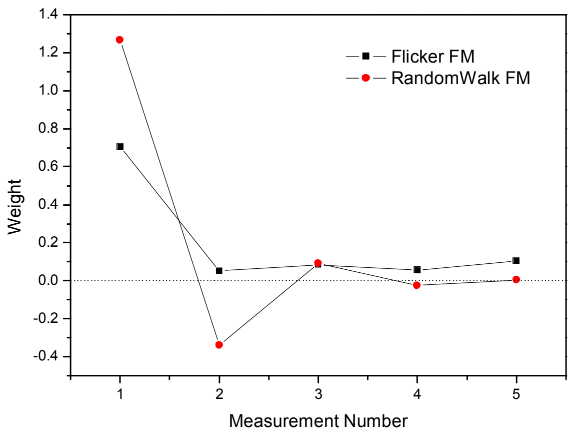

Figure 3. Weights when five measurements are used. Measurement number 1 is the most recent measurement.

FM. Weights for the most recent one and the oldest one is 0.704 and 0.105, respectively. For random-walk FM, the weights show negative values for even-indexed measurements. This means that when we estimate the frequency, even indexed frequencies are to be negatively reflected in predicted frequency.

We have so far considered the timescale with only one noise type. Real clock, however, shows mixed noise characteristics in most cases. Theory in [1] can also be used for a timescale with mixed noise types.

Fig. 4 shows the ratio of the uncertainty to the Allan deviation at a sampling time of measurement interval, for various numbers of measurements used in prediction, for mixed noise type case. The noise characteristics, $\sigma_y(\tau)$, of the time scale are assumed to be defined in terms of the three basic noise types (white FM, flicker FM, and random-walk FM). Allan deviations of the three noise types have the same value at τ sampling time, same as the measurement interval. In this case, all three different noise types will contribute to the uncertainty and hence the determination of the weights. As shown in Fig. 4, the uncertainty of using only one measurement shows only 1.6 % larger value of the achievable minimum uncertainty. With two measurements, it becomes 0.3 %. This is due to the random-walk FM and flicker FM noise types.

Weights corresponding to Fig. 4 when we use five measurements are shown in Fig. 5. The most recent measurement shows weight of 0.83, which is larger than the case for white FM and flicker FM nose. This large weight compared to those of the rest of the measurements can be explained as the 'pulling up' the first weight and negatively 'pulling down' the second weight by random-walk FM noise. Slight increase in the weight of the oldest measurement is likewise due to the flicker-FM noise.

We can infer that if the measurement and prediction are done for a much shorter time interval than τ, the uncertainty and weights will follow the characteristic of white FM case, resulting in almost same weights for all the measured frequencies. In other extreme case when the measurement interval is much longer than τ, the random-walk FM noise dominates and we may need to consider only two or three previous measurements to get the minimum uncertainty frequency prediction.

B. *For a timescale with drift*

We have so far assumed a timescale without drift. However, actual frequency standard can have non-negligible drift. Typical example is a commercial hydrogen maser, which has an intrinsic drift. Now we assume that a timescale has drift and the drift rate is already known to us. Even though the actual determination of the drift rate can have unavoidable uncertainty, we will focus only on the uncertainty due to the estimation process and neglect the uncertainty in estimating the drift rate.

As shown in Fig. 6, since the drift rate is known, it is easy to remove the drift from the measured data. We can apply the theory to the drift removed data as explained in the previous section. We will add the drift rate to the predicted frequency with drift removed data to get the final result with drift.

The phase of the drifting timescale can be expressed as follows.

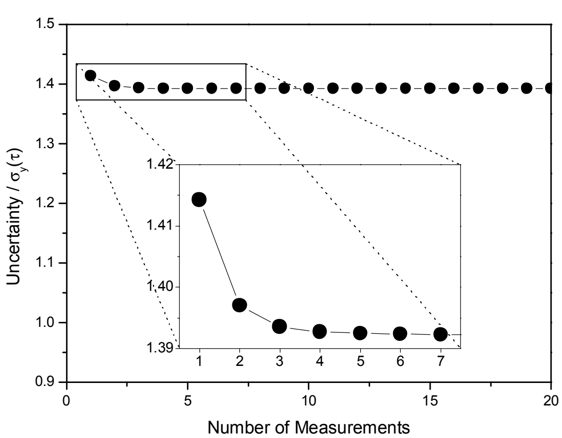

Figure 4. Uncertainty to Allan devialtion ratio vs. number of measurements used for mixed noise type.

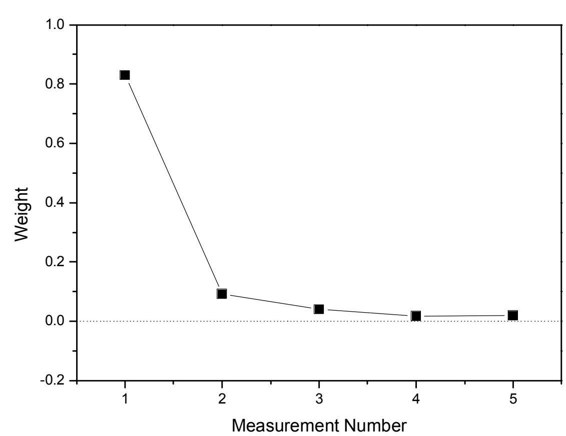

Figure 5. Weights when five measurements are used for mixed noise type. Measurement number 1 is the most recent measurement.

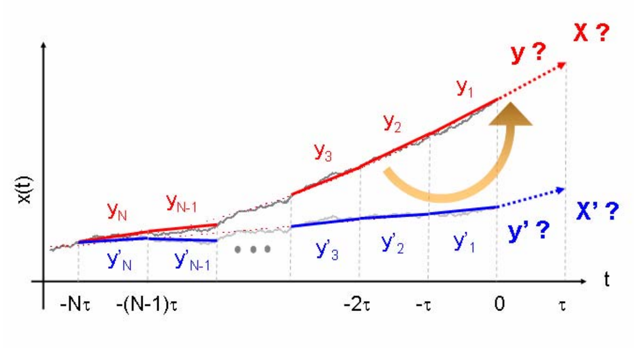

Figure 6. Frequency estimation when the timescale has a known drift rate.

$$x(t) = x_0 + f \cdot t + \frac{1}{2} D \cdot t^2 + \text{noise}, \tag{6}$$

where, f is a frequency and D is the drift rate. If we remove the known drift from the phase, we can get the phase without drift.

$$x'(t) = x(t) - \frac{1}{2} D \cdot t^2 = x_0 + f \cdot t + +\text{noise} \cdot \tag{7}$$

From the phase data, we can calculate the drift removed frequency, y', for each measurement time interval.

$$\begin{aligned} y'_n &= \frac{1}{\tau}\left(x'(-(n-1)\tau) - x'(-n\tau)\right) \\ &= y_n + D \cdot (n - \frac{1}{2}) \cdot \tau \end{aligned} \tag{8}$$

Where, n is the measurement index as shown in Fig. 6. As shown in (8), the drift removed frequency is the sum of the actual frequency and the correction term due to the drift.

Now, we have only drift removed frequencies, we can apply the theory and calculate the uncertainty and optimum weights as shown in the previous section. A predicted frequency without drift can then be calculated using the obtained optimum weights. It can be expressed by the weighted sum of drift biased frequencies and additive correction term as follows.

$$\begin{aligned} y' &= \sum_{n=1}^{N} a_n y'_n \\ &= \sum_{n=1}^{N} a_n y_n + D \cdot \tau \sum_{n=1}^{N} a_n (n - \frac{1}{2}) \end{aligned} \tag{9}$$

Finally we can get a frequency estimate for the drifting timescale considering the drift as follows.

$$\begin{aligned} y &= y' + \frac{1}{2} D \cdot \tau \\ &= \sum_{n=1}^{N} a_n y_n + D \cdot \tau \sum_{n=1}^{N} a_n n \end{aligned} \tag{10}$$

For the case considered here using previous consecutive measurements, we can get a predicted frequency by calculating the weighted average of the measured frequencies as if there is no drift and then adding the simple correction term. Since we assume that the 'systematic' drift can be removed from the data without adding uncertainty, the uncertainty of the prediction is the same as the case without drift as long as the noise characteristics are the same.

IV. Conclusion

We presented a method for current frequency estimation with minimum uncertainty using the past N consecutive measurements. We found that only three previous measurements were enough to reduce the uncertainty to less than 5 % over the achievable minimum uncertainty for flicker FM. For white FM noise, we needed to use 10 previous measurements to reach the same result. Even remarkable case was shown for a random-walk FM noise type, the uncertainty became less than 1 % over the limit value with only two measurements. The result with mixed noise type case was also analyzed. For a drifting timescale, simple method to predict the frequency and phase was proposed.

References

[1] D. H. Yu, M. Weiss, and T.E.Parker, "Uncertainty of a frequency comparison with distributed dead time and measurement interval offset," Metrologia, vol. 44, pp. 91-96, 2007.

[2] R. J. Douglas and J. -S. Boulanger, "Standard Uncertainty for Average Frequency Traceability," Proc. 11th European Frequency Time and Forum, pp. 345-349, 1997.